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The degree of irrationality of hypersurfaces in various Fano varieties

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Abstract. The purpose of this paper is to compute the degree of irrationality of hypersurfaces of sufficiently high degree in various Fano varieties: quadrics, Grassmannians, products of projective space, cubic threefolds, cubic fourfolds, and complete intersection threefolds of type (2,2). This extends the techniques of Bastianelli, De Poi, Ein, Lazarsfeld, and the second author who computed the degree of irrationality of hypersurfaces of sufficiently high degree in projective space. A theme in the paper is that the fibers of low degree rational maps from the hypersurfaces to projective space tend to lie on curves of low degree contained in the Fano varieties. This allows us to study these maps by studying the geometry of curves in these Fano varieties.

Introduction

The *degree of irrationality* of an n-dimensional algebraic variety X, denoted $\mathrm{irr}(X)$, is the minimal degree of a dominant rational map

$$\phi\colon X \dashrightarrow \mathbf{P}^n.$$

The aim of this paper is to compute the degree of irrationality of hypersurfaces in various Fano varieties: quadrics, cubic threefolds, cubic fourfolds, complete intersection threefolds of type (2,2), Grassmannians, and products of projective spaces. Throughout we work with varieties over $\mathbb{C}$.

Recently there has been a great deal of interest in understanding different measures of irrationality of higher dimensional varieties. Bastianelli, Cortini, and De Poi conjectured ([1, Conj. 1.5]) that if X is a very general d hypersurface

$$X = X_d \subset \mathbf{P}^{n+1}$$

with $d \geq 2n + 1$, then $\mathrm{irr}(X) = d - 1$. They proved their conjecture in the case X is a surface or threefold. This conjecture was proved in full by Bastianelli, De Poi, Ein, Lazarsfeld, and the second author in [4]. Gounelas and Kouvidakis [7] computed the covering gonality and the degree of irrationality of the Fano surface of a generic cubic threefold. Bastianelli, Ciliberto, Flamini, and Supino [2] computed

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the covering gonality of a very general hypersurface in $\mathbf{P}^{n+1}$. Recently, Voisin [13] proved that the covering gonality of a very general n-dimensional abelian variety goes to infinity with n.

In this paper we show that the ideas in the proof of [4, Thm. C] can be extended to compute the degree of irrationality of hypersurfaces in many Fano varieties. For example, let $\mathbf{Q} \subset \mathbf{P}^{n+2}$ be a smooth quadric in projective space.

Theorem A. *Let*

$$X = X_d \subset \mathbf{Q} \subset \mathbf{P}^{n+2}$$

be a very general hypersurface in $\mathbf{Q}$ *with* $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$. *If* $d \geq 2n$ *then* $\mathrm{irr}(X) = d$.

We have other results for hypersurfaces in cubic threefolds and cubic fourfolds.

Theorem B. *Let*

$$X = X_d \subset Z \subset \mathbf{P}^{n+2}$$

be a smooth complete intersection of type $(3, d)$ *in a smooth cubic hypersurface.*

(1) *If* $n = 2$ *and* $d \geq 8$ *then*

$$\mathrm{irr}(X) = \begin{cases} 2d-2 \text{ if } X \text{ contains a line,} \\ 2d \text{ otherwise,} \end{cases}$$

and any rational map $X \dashrightarrow \mathbf{P}^2$ *with degree equal to* $\mathrm{irr}(X)$ *is birationally equivalent to projection from a line in* Z.

(2) *If* $n = 3$, $d \geq 13$ *and* X *is very general in* $|\mathcal{O}_Z(d)|$, *then* $\mathrm{irr}(X) = 2d$.

Now let $Z = Z_{(2,2)} \subset \mathbf{P}^5$ be a smooth complete intersection of two quadrics.

Theorem C. *Let*

$$X = X_d \subset Z$$

be a smooth surface in Z *with* $X \in |\mathcal{O}_Z(d)|$. *If* $d \geq 8$ *then*

$$\mathrm{irr}(X) = \begin{cases} 2d-2 \text{ if } X \text{ contains a plane conic,} \\ 2d-1 \text{ if } X \text{ contains a line and no conic,} \\ 2d \text{ otherwise.} \end{cases}$$

Moreover, any rational map $X \dashrightarrow \mathbf{P}^n$ *with degree equal to* $\mathrm{irr}(X)$ *is birationally equivalent to the projection from a plane contained in one of the quadrics in the linear series* $|I_Z(2)|$.

Furthermore, we compute the degree of irrationality of hypersurfaces in Grassmannians. Let

$$\mathbf{G} = \mathrm{Gr}(k, m) \subset \mathbf{P}$$

be the Grassmannian of k planes in an m dimensional vector space embedded via its Plücker embedding. Assume $k \neq 1, m-1$ (the excluded cases are covered by [4, Thm. C]).

Theorem D. *Let*

$$X = X_d \subset \mathbf{G}$$

be a very general hypersurface with $X \in |\mathcal{O}_{\mathbf{G}}(d)|$. *If* $d \geq 3m - 5$ *then* $\mathrm{irr}(X) = d$.

Finally, let $\mathbf{P} = \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_k}$ be a product of k projective spaces with $k \geq 2$.

Theorem E. *Let*

$$X = X_{d_1,\ldots,d_k} \subset \mathbf{P}$$

be a very general hypersurface with $X \in |\mathcal{O}_{\mathbf{P}}(d_1, \ldots, d_k)|$. *Let* p *be the minimum of* $\{d_i - m_i - 1\}$. *If* $p \geq \max\{m_i\}$ *then* $\mathrm{irr}(X) = \min\{d_i\}$.

A recurring theme throughout the paper is that the positivity of the canonical linear series helps to control the degree of irrationality. For example, given a dominant rational map:

$$\phi \colon X \dashrightarrow \mathbf{P}^n,$$

every finite fiber of ϕ satisfies the *Cayley-Bacharach condition* (Definition 1.6) with respect to the canonical linear series $|\omega_X|$. This affects the possible projective configurations of the fibers. As a consequence, if $Z \subset \mathbf{P}$ is one of the Fano varieties above in its natural projective embedding, and $X \subset Z$ is a hypersurface of sufficiently high degree, then we will see that any fiber of ϕ must lie on a low degree curve $C \subset Z$ (in the cases we consider, C will always have degree ≤ 2).

This allows us to study low degree maps to $\mathbf{P}^n$ by studying the geometry of low degree curves on these Fano varieties. In some cases (when Z is a cubic threefold, or a (2,2) complete intersection threefold), the geometry of the spaces parametrizing low degree curves is explicit enough to complete the computation of the degree of irrationality of $X \subset Z$. In the other cases, we use the assumption that X is very general and follow the ideas of [4, Prop. 3.8] to complete the proofs.

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1. Background

In this section we introduce the main definitions, and recall some known results. There is a nice introduction to these ideas in [4], and we refer the interested reader there for more details. At the end of this section, we also prove a preliminary result about points in projective space satisfying the Cayley–Bacharach condition.

Definition 1.1. Let X be an n-dimensional algebraic variety. The *degree of irrationality of* X, denoted $\mathrm{irr}(X)$, is the minimal degree of a rational map

$$\phi\colon X \dashrightarrow \mathbf{P}^n.$$

The degree of irrationality of X is a birational invariant of X. It is possible to give lower bounds on $\mathrm{irr}(X)$ by understanding the birational positivity of K_X, in an appropriate sense.

Definition 1.2. Let L be a line bundle on a variety X.

(1) We say L is *p-very ample* if for all 0-dimensional subschemes $\xi \subset X$ of length $p+1$, the restriction map

$$H^0(X, L) \to H^0(X, L|_\xi)$$

is surjective.

(2) We say L *satisfies property* $(\mathrm{BVA})_p$ if there exists a nonempty open set $\emptyset \neq U \subset X$ such that for all 0-dimensional subschemes $\xi \subset U$ of length $p+1$, the restriction map

$$H^0(X, L) \to H^0(X, L|_\xi)$$

is surjective.

Example 1.3. A line bundle L satisfies $(\mathrm{BVA})_0$ if and only if L is effective. Moreover, L satisfies $(\mathrm{BVA})_1$ if and only if the linear series $|L|$ maps X birationally onto its image in projective space.

Example 1.4. If $L = \mathcal{O}_{\mathbf{P}}(p)$, then L is *p-very ample* and in particular, L satisfies $(\mathrm{BVA})_p$.

Theorem 1.5 ([4, Thm. 1.10]). *Let X be a smooth projective variety and suppose that ω_X satisfies property $(\mathrm{BVA})_p$, then*

$$\mathrm{irr}(X) \geq p+2.$$

One fundamental fact that we will use is that the fibers of a dominant rational map

$$\phi\colon X \dashrightarrow \mathbf{P}^n$$

lie in special position, in the sense that they satisfy the *Cayley–Bacharach condition*.

Definition 1.6. Let $\mathcal{S} \subset \mathbf{P}$ be a finite subset of projective space. We say that the set $\mathcal{S}$ *satisfies the Cayley–Bacharach condition with respect to* $|\mathcal{O}_{\mathbf{P}}(m)|$ (or just $\mathcal{S}$ *satisfies* $\mathrm{CB}(m)$) if any divisor $D \in |\mathcal{O}_{\mathbf{P}}(m)|$ which contains all but one point of $\mathcal{S}$, contains all of $\mathcal{S}$.

Let $X \subset \mathbf{P}$ be a smooth n-dimensional subvariety of projective space. Assume that $\omega_X = \mathcal{O}_X(m)$ for some m. The following proposition was proven by Bastianelli, Cortini, and De Poi.

Proposition 1.7 ([1, Prop. 2.3]).

(1) Assume that $\Gamma \subset X \times \mathbf{P}^n$ is a reduced subscheme of pure dimension n. Assume that $y \in \mathbf{P}^n$ is a smooth point for the projection

$$\pi_2|_\Gamma : \Gamma \to \mathbf{P}^n.$$

Then the set $\mathcal{S} = \pi_1((\pi_2|_\Gamma)^{-1}(y))$ satisfies $\mathrm{CB}(m)$.

(2) In the special case when Γ is the graph of a rational map $\phi \colon X \dashrightarrow \mathbf{P}^n$, (1) implies that a general fiber of ϕ satisfies $\mathrm{CB}(m)$.

Furthermore, those authors show that there are strong geometric consequences imposed on small sets $\mathcal{S} \subset \mathbf{P}$ which satisfy $\mathrm{CB}(m)$.

Lemma 1.8. ([1, Lem. 2.4]). *Let $n \geq 2$ and let $\mathcal{S} \subset \mathbf{P}$ be a set of r points in projective space which satisfy* $\mathrm{CB}(m)$. *Then $r \geq m + 2$. Moreover, if $r \leq 2m + 1$ then all the points in $\mathcal{S}$ lie on a line $\ell \subset \mathbf{P}^n$.*

In order to prove Theorems B and C we need a mild generalization of Lemma 1.8. We encourage the casual reader to skip the proof of the following theorem.

Theorem 1.9. *Let $\mathcal{S}$ be a set of r points in projective space which satisfy* $\mathrm{CB}(m)$. *If*

$$r \leq (5/2)m + 1$$

then $\mathcal{S}$ is contained in a curve C with $\deg(C) \leq 2$ *(either a line, a plane conic, or a union of two lines).*

To prove Theorem 1.9, we start with the case when $\mathcal{S} \subset \mathbf{P}^2$ is contained in a plane.

Lemma 1.10. *Let $\mathcal{S} \subset \mathbf{P}^2$ be a set of r points which satisfy* $\mathrm{CB}(m)$. *If*

$$r \leq (5/2)m + 1$$

then $\mathcal{S}$ is contained in a curve $C \subset \mathbf{P}^2$ with degree ≤ 2.

Proof. We proceed by induction. First we need to take care of all cases when $m \leq 3$.

When $m = 1$, then $r \leq 3$, so there is a conic containing all points in $\mathcal{S}$. When $m = 2$, then $r \leq 6$. There must be a conic C through 5 of the points in $\mathcal{S}$, and because $\mathcal{S}$ satisfies $\mathrm{CB}(2)$ we know $\mathcal{S} \subset C$.

Let $m = 3$ and first assume there is a line containing $\rho \geq 3$ points. The remaining $r - \rho$ points satisfy $\mathrm{CB}(2)$ and thus lie on a line by Lemma 1.8, so $\mathcal{S}$ is contained in the union of 2 lines. Now assume no 3 points lie on a line, and take a conic C which contains ρ points where $\rho \geq 5$. Thus there are $r - \rho \leq 3$ remaining points. These points satisfy $\mathrm{CB}(1)$. Thus by Lemma 1.8, we can conclude that $\rho = 8$ and thus all the points of $\mathcal{S}$ must lie on the conic C.

Proceeding by induction, let

$$C_1 \text{ be either } \begin{cases} 1. \text{ a line in } \mathbf{P}^2 \text{ such that } \#C_1 \cap \mathcal{S} = \rho \geq 3, \text{ or} \\ 2. \text{ a conic in } \mathbf{P}^2 \text{ such that } \#C_1 \cap \mathcal{S} = \rho \geq 5. \end{cases}$$

In case 1, the remaining $r - \rho$ points satisfy $\mathrm{CB}(m-1)$, and

$$r - \rho \leq (5/2)(m-1) + 1.$$

In case 2, the remaining $r - \rho$ points satisfy $\mathrm{CB}(m-2)$ and

$$r - \rho \leq (5/2)(m-2) + 1.$$

In either case, by induction, there is a curve $D \subset \mathbf{P}^2$ containing $\mathcal{S}$ which is the union of lines and conics and satisfies $\deg(D) \leq 4$. Moreover, if C_1 is a line then $\deg(D) \leq 3$.

Now assume that D contains a line $C_1 \subset D$ and C_1 contains a point in $\mathcal{S}$ which is not contained in any other component of D. Then the points on C_1 which don't lie on another component of D satisfy $\mathrm{CB}(m-3)$, thus as $m \geq 4$ by Lemma 1.8 there are at least 3 points on C_1. Thus by the previous paragraph we can assume that $\deg(D) \leq 3$. So we are in the situation where all the points are on a line C_1 and a conic C_2 and $D = C_1 \cup C_2$. Suppose there are a total of ρ' points which do not lie on C_1. These points satisfy $\mathrm{CB}(m-1)$. If $\rho' \leq 2(m-1)+1 = 2m-1$ then we are done by Lemma 1.8. So assume for contradiction that

$$\rho' \geq 2m.$$

Returning to the points on C_1 we see there are at most $r - \rho'$ points on C_1 not contained in C_2. These points satisfy $\mathrm{CB}(m-2)$, therefore by Lemma 1.8

$$m \leq r - \rho'.$$

However, the previous inequality implies

$$r - \rho' \leq r - 2m \leq (5/2)m + 1 - 2m \leq m/2 + 1.$$

The right hand side is less than m (for $m \geq 4$), which gives the contradiction.

Now we can assume that $\mathcal{S} \subset D = C_1 \cup C_2$ is the union of two smooth conics. Assume without loss of generality that C_1 contains at least half (but not all) of the points in $\mathcal{S}$. If ρ is the number of points in $\mathcal{S}$ which are not contained in C_1, then

$$\rho \leq \lfloor r/2 \rfloor \leq \lfloor (5/4)m + 1/2 \rfloor.$$

Moreover, these ρ points satisfy $\mathrm{CB}(m-2)$, and when $m \geq 4$:

$$\rho \leq \lfloor (5/4)m + 1/2 \rfloor \leq 2(m-2) + 1.$$

Therefore, by Lemma 1.8 we know there are at least 4 points on a line so we are done by the first case. □

Proof of Theorem 1.9. By Lemma 1.10, we can assume that the points $\mathcal{S}$ are not contained in a plane. Again we plan to proceed by induction, and we need to start by checking the cases $m = 1$ and 2. We leave the case $m = 1$ to the reader.

Assume $m = 2$, so $r \leq 6$. Let ρ (respectively σ) be the maximum number of points in $\mathcal{S}$ contained in a plane Λ (respectively a line). Then by assumption we have $r > \rho$, $\rho \geq \sigma + 1$, and $\rho \geq 3$. The $r - \rho$ points in the complement of the plane satisfy CB(1). Combining inequalities and applying Lemma 1.8 we have $r-\rho = 3$, and all these points lie on a line. This is a contradiction as the inequalities imply

$$3 \leq \sigma \leq \rho - 1 \leq 2.$$

Thus in this case, all the points in $\mathcal{S}$ lie on a plane.

Now assume $\rho \geq 3$. Again let ρ be the maximum number of points in $\mathcal{S}$ which are contained in a plane. The remaining $r-\rho$ points satisfy CB(m-1), and we have the inequality

$$r - \rho \leq (5/2)(m-1) + 1.$$

So by induction the remaining points lie on a plane conic or on a pair of skew lines.

In the case when the remaining points lie on a plane conic, by the definition of ρ we know that $r-\rho \leq \rho$, and thus

$$r - \rho \leq r/2 \leq (5/4)m + 1/2.$$

As $m \geq 3$ we have that

$$(5/4)m + 1/2 \leq 2(m-1) + 1$$

and by Lemma 1.8 we have that all the $r-\rho$ points lie on a line C_1. By Lemma 1.8 there are at least $m+1$ points of $\mathcal{S}$ on C_1. Therefore, there are at most $r-m-1$ points not on C_1 and these points satisfy CB$(m-1)$. Combining our inequalities we have

$$r - m - 1 \leq (5/2)m + 1 - m - 1 \leq (3/2)m \leq 2(m-1) + 1.$$

Thus by Lemma 1.8 the points which are not contained on C_1 are contained in another line C_2, which proves the first case.

The last case to take care of is when $m \geq 3$ and the remaining $r-\rho$ points lie on two skew lines C_1 and C_2. Suppose there are ρ' points on C_1 and ρ'' points on C_2 (and thus $r-\rho = \rho' + \rho''$). In this case the ρ' points on C_1 and the ρ'' points on C_2 both satisfy CB(m-2). Thus by Lemma 1.8 we have, $\rho', \rho'' \geq m$. Moreover, by the definition of ρ it is clear that $\rho \geq \max\{\rho', \rho''\} \geq m$ (points on a line lie on a plane). Therefore, we have

$$3m \leq \rho + \rho' + \rho'' = r \leq (5/2)m + 1$$

which is a contradiction. □

Remark 1.11. In the setting of Theorem 1.9, if the set $\mathcal{S} \subset \ell_1 \cup \ell_2$ is contained in the union of two lines then one can show that each line contains at least $m+1$ points. Indeed, the complement of the points in ℓ_1 satisfies CB$(m-1)$, thus by Lemma 1.8 there are at least $m+1$ points in ℓ_2.

2. Quadrics

Let $\mathbf{Q} \subset \mathbf{P}^{n+2}$ be a smooth $(n+1)$-dimensional quadric in projective space. The aim of this section is to prove Theorem A, that is if $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$ is a very general hypersurface with $n \geq 1$ and $d \geq 2n$, then $\operatorname{irr}(X) = d$.

Example 2.1. When $n = 1$, i.e. $X \subset \mathbf{Q} \subset \mathbf{P}^3$ is a curve in a smooth quadric in $\mathbf{P}^3$, then we know $\mathbf{Q} \cong \mathbf{P}^1 \times \mathbf{P}^1$. Projection onto either factor

$$\phi_0 \colon X \to \mathbf{P}^1$$

gives a degree d map to $\mathbf{P}^1$. By adjunction $\omega_X = \mathcal{O}_X(d-2)$ is $(d-2)$-very ample. By Theorem 1.5

$$d \geq \operatorname{irr}(X) = \operatorname{gon}(X) \geq d - 2 + 2 = d.$$

In higher dimensions, this can be generalized as it is always possible to project from a line $\ell \subset \mathbf{Q}$

Proposition 2.2. *If $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$ is any hypersurface, then there exists*

$$\phi_0 \colon X \dashrightarrow \mathbf{P}^n$$

such that $\deg(\phi_0) = d$. *Therefore* $\operatorname{irr}(X) \leq d$.

Proof. Choose a line $\ell \subset \mathbf{Q}$ which meets X properly. Let $p_\ell \colon \mathbf{P}^{n+2} \dashrightarrow \mathbf{P}^n$ denote the linear projection from ℓ and set $\phi_0 = p_\ell|_X$. The closure of each fiber of p_ℓ is a plane $\mathbf{P}^2 \subset \mathbf{P}^{n+2}$ containing ℓ. Thus, for a general such plane we can compute

$$\begin{aligned} \deg(\phi_0) &= \left(\text{length of } \mathbf{P}^2 \cap X\right) - \left(\text{length of } \mathbf{P}^2 \cap X \text{ supported on } \operatorname{Bs}(\phi_0)\right) \\ &= \operatorname{length}(\mathbf{P}^2 \cap X) - \operatorname{length}(\ell \cap X) = 2d - d = d \end{aligned}$$

□

In the case of [4, Thm. C], those authors prove that if $X \subset \mathbf{P}^{n+1}$ is a very general hypersurface with $d \geq 2n+2$, then any degree $d-1$ map is given by projection from a point up to postcomposition with a birational automorphism of $\mathbf{P}^n$. Such a simple description is not possible for quadrics in all dimensions. Already in Example 2.1 we see that there are two possible projections $X \to \mathbf{P}^1$ (though both projections are still given by projection from some line in $\mathbf{Q}$). So maps computing the gonality are not unique already when $n = 1$. In the next example, we show that when n is odd the degree of irrationality is not only realized by projection from a line.

Example 2.3. (Another map realizing $\operatorname{irr}(X)$ when n is odd.). Let $X \subset \mathbf{Q} \subset \mathbf{P}^{n+2}$ be as above and assume that $n = 2k-1$ is odd. There exist non-intersecting linear subspaces of dimension k:

$$\mathbf{P}(V), \mathbf{P}(W) \subset \mathbf{Q}.$$

These are *maximal isotropic subspaces* which are in the same family. As they do not intersect we may write $\mathbf{P}^{n+2} = \mathbf{P}(V \oplus W)$.

The rational map

$$p_{V,W} \colon \mathbf{P}(V \oplus W) \dashrightarrow \mathbf{P}(V) \times \mathbf{P}(W).$$
$$[v \oplus w] \mapsto [v] \times [w].$$

maps $\mathbf{Q}$ onto a rational divisor $B \subset \mathbf{P}(V) \times \mathbf{P}(W)$ of type $(1, 1)$, and contracts lines in $\mathbf{Q}$ of the form

$$\ell = \{[sv \oplus tw] | [s : t] \in \mathbf{P}^1\}.$$

The restriction $\phi_1 = p_{V,W}|_X$ has degree d if V and W are chosen generally.

In order to reach a contradiction and prove Theorem A we assume that there exists a map

$$\phi \colon X \dashrightarrow \mathbf{P}^n$$

with $\delta = \deg(\phi) < d$. First, we note that all fibers of ϕ must lie on a line $\ell \subset \mathbf{Q}$.

Lemma 2.4. *If $d \geq 2n$ and $(d, n) \neq (2, 1)$ then a general fiber of ϕ lies on a line $\ell \subset \mathbf{P}^{n+2}$ which is contained in $\mathbf{Q}$.*

Proof. By adjunction $\omega_X = \mathcal{O}_X(d - n - 1)$. The assumption that $d \geq 2n$ implies that $\delta \leq 2(d - n - 1) + 1$. Thus as a general fiber of ϕ satisfies Cayley–Bacharach with respect to $|\omega_X|$ by Lemma 1.8 a general fiber of ϕ must lie on a line $\ell \subset \mathbf{P}^{n+2}$. And assuming $(d, n) \neq (2, 1)$, by Theorem 1.5 at least 3 points lie on the line, so as a consequence of Bezout's theorem we have $\ell \subset \mathbf{Q}$. □

By the previous lemma, a general point $y \in \mathbf{P}^n$ parameterizes a line $\ell_y \subset \mathbf{Q}$ (the span of the fiber $\phi^{-1}(y)$). This induces a rational map $\mathbf{P}^n \dashrightarrow \mathrm{Fano}(\mathbf{Q})$, where $\mathrm{Fano}(\mathbf{Q})$ is the Fano variety of lines contained in $\mathbf{Q}$ (the *orthogonal Grassmannian*). Resolving the map gives

$$f \colon B \to \mathrm{Fano}(\mathbf{Q})$$

where B is a smooth and rational projective variety. The map f gives rise to the following fundamental diagram whose terms are defined below:

$$\begin{tikzcd} X \arrow[d, hook] & X' \arrow[l] \arrow[d, hook] & \\ \mathbf{Q} & F \arrow[l, "\pi"'] \arrow[d, "\psi"] & \\ & B \arrow[r, "f"] & \mathrm{Fano}(\mathbf{Q}). \end{tikzcd} \tag{1}$$

Here $\psi \colon F \to B$ is the $\mathbf{P}^1$-bundle defined as the pullback of the natural $\mathbf{P}^1$-bundle over $\mathrm{Fano}(\mathbf{Q})$. Thus F comes with a natural projection $\pi \colon F \to \mathbf{Q}$. The fact that X

is not uniruled implies π is generically finite. To define X' consider the rational map:

$$\mathrm{id}_X \times \varphi\colon X \dashrightarrow X \times B \subset \mathbf{Q} \times B$$

which is the graph of the rational map φ. The image of $\mathrm{id}_X \times \varphi$ is contained in F. Set

$$X' := \overline{\mathrm{Image}(\mathrm{id}_X \times \varphi)},$$

i.e. let X' be the closure of the image of the graph of φ.

Lemma 2.5. *If $d \geq 2n$ and $(d, n) \neq (2, 1)$ then the map π in (1) is birational. In particular f determines a "congruence of order one" on $\mathbf{Q}$.*

Proof. The proof is identical to the proof given in [1, Thm. 4.3], but we give the argument here for the convenience of the reader. Note that $\pi_*(\pi^*([X])) = \deg(\pi)[X]$, so it suffices to show that $\pi_*(\pi^*([X])) = [X]$. We have that $X' \subset \pi^{-1}(X)$, so there are irreducible divisors E_i on F so that:

$$\pi^*[X] = a[X'] + \sum a_i[E_i],$$

with $a, a_i > 0$. As $\pi_*([X']) = [X]$, it suffices to show that $a = 1$ and $\pi_*([E_i]) = 0$.

To prove $a = 1$ we remark that for a fiber $\ell \subset F$ of ψ we have $\pi^*([X])\cdot[\ell] = d$, $[X']\cdot[\ell] = \deg(\phi)$, and $[E_i]\cdot[\ell] \geq 0$. By Theorem 1.5 we have $\deg(\phi) \geq d-n+1$. It follows from the assumption $d \geq 2n$ that $a = 1$.

Assume for contradiction that $\pi_*([E_i]) \neq 0$. Then E_i dominates X. If $\psi_*([E_i]) = 0$ then the fibers of $\psi|_{E_i}$ must be positive dimensional and thus the fibers are lines. This implies E_i is ruled, which in turn implies that X is uniruled, which is absurd as X is general type. So we can assume $\pi_*([E_i]) \neq 0$. Then E_i gives a correspondence between X and B. By Proposition 1.7 we have that for a general point $y \in B$, the set $\mathcal{S} = \pi((\psi|_{E_i})^{-1}(y))$ satisfies $\mathrm{CB}(d-n-1)$. It follows from Proposition 1.8 that $\#\mathcal{S} \geq d-n+1$ and thus if ℓ is a fiber of ψ we have $[E_i]\cdot[\ell] \geq d-n+1$. Using the previous paragraph is follows that

$$d = \pi^*([X])\cdot[\ell] \geq [X']\cdot[\ell] + [E_i]\cdot[\ell] \geq 2d-2n+1 \geq d+1.$$

which is a contradiction. ◻

Proof of Theorem A. We follow the proof of [4, Thm. C]. Assume for contradiction that there exists a dominant rational map

$$\phi\colon X \dashrightarrow \mathbf{P}^n$$

with $\delta = \deg(\phi) < d$. By Theorem 1.5 we know $\delta \geq d-n+1$, and thus by our assumption on δ we can assume $n \geq 2$. Now consider the divisor

$$\pi^*X = X' + \sum a_i E_i.$$

By Lemma 2.5, the map π is birational so we conclude that $\pi_* E_i = 0$. As the E_i are effective divisors, for a fiber ℓ of ψ we must have $E_i \cdot [\ell] \geq 0$. We also know

$$X' \cdot [\ell] = \deg(\phi) = \delta \text{ and } \pi^* X \cdot [\ell] = d.$$

The lower bound on δ implies there exists $E = E_i$ with

$$0 < c = \deg(\psi|_E) = E \cdot [\ell] \leq d - \delta \leq n - 1.$$

The above calculation implies that E intersects every fiber ℓ. By Lemma 2.5, the images of ℓ under π sweep out $\mathbf{Q}$. Thus every point in $\mathbf{Q}$ is connected to $\pi(E)$ by a line inside $\mathbf{Q}$. The dimension of lines through a single point in $\mathbf{Q}$ is $n-1$. It follows that $\dim(\pi(E)) \geq 1$.

Thus the image $\pi(E)$ has covering gonality $\leq n-1$. By Lemma 2.6 (which we prove below) we have

$$c \geq e + d - 2n + 1. \tag{2}$$

There is another inequality relating e and c which arises from understanding the contribution of E to the effective divisor $K_{F/\mathbf{Q}}$. As shown in [4, Cor. A.6] we have

$$\mathrm{ord}_E K_{F/\mathbf{Q}} \geq n - e.$$

Moreover,

$$-2 = K_F \cdot [\ell] = (K_{F/\mathbf{Q}} + \pi^* K_{\mathbf{Q}}) \cdot [\ell] = (K_{F/\mathbf{Q}} \cdot [\ell]) - n - 1.$$

Thus

$$n - 1 = K_{F/\mathbf{Q}} \cdot [\ell] \geq \mathrm{ord}_E(K_{F/\mathbf{Q}}) E \cdot [\ell] \geq (n-e)c. \tag{3}$$

Now combining Eqs. (2) and (3) we get

$$\frac{n-1}{n-e} \geq c \geq e + d - 2n + 1.$$

Rearranging, we have

$$2n - 1 + e\left(\frac{n-1}{(n-e)e} - 1\right) \geq d.$$

The left hand side is strictly less than $2n$ which contradicts our assumption on the degree of X. □

To complete the above we need the following lemma.

Lemma 2.6. *Let $X = X_d \subset \mathbf{Q}$ be a very general hypersurface as in Theorem A. Suppose that $W \subset X$ is a subvariety of X of dimension e and covering gonality c. Then we have*

$$c \geq e + d - 2n + 1.$$

Remark 2.7. The arguments in the proof of Lemma 2.6 are due to Ein and Voisin [6,12], and closely follows the proof of [4, Prop. 3.8]. Indeed, one could directly apply [4, Prop. 3.8] in the above proof if we assume slightly weaker bounds on the degree of X. However, we include the proof of this lemma for the convenience of the reader, and because we will use some of the results we prove in later sections.

Let $\mathbf{Y} \subset \mathbf{P}^M$ be a rational homogeneous space (for example a smooth quadric) embedded in projective space via $|\mathcal{O}_{\mathbf{Y}}(1)|$, and let L be another very ample line bundle on $\mathbf{Y}$. Let $U \subset H^0(\mathbf{Y}, L)$ be the open subset of which parametrizes smooth divisors. Then there is a universal smooth divisor $\mathcal{X}$ over U

$$\begin{tikzcd} \mathcal{X} \subset U \times \mathbf{Y} \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbf{Y}. \\ U & \end{tikzcd}$$

In this setting we have the following proposition.

Proposition 2.8. *Let $X \in |L|$ be a very general divisor, and suppose that X contains a subvariety of dimension e and covering gonality c. Let $n = \dim(X)$ and assume that $\omega_X(-n)$ is p-very ample. If $T_{\mathcal{X}}(1)|_X$ is globally generated then*

$$c \geq e + p + 2.$$

Proof. We have assumed that a very general $X \in |L|$ contains an e-dimensional subvariety $S' \subset X$ which is swept out by curves of gonality c. By a standard argument, S' exists in a family over U, that is we have a diagram:

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[dr] \arrow[rr, bend left, "f"] & \mathcal{S}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "p_1"] \\ & V \arrow[r] & U \end{tikzcd}$$

where $V \to U$ is étale, $\mathcal{S}' \subset \mathcal{X} \times_U V$ is a family of e-dimensional subvarieties of $\mathcal{X} \times_U V$ which are swept out by curves with gonality c, and $\mathcal{S} \to \mathcal{S}'$ is a resolution of the total space of $\mathcal{S}'$. In particular, after shrinking V we can assume that $\mathcal{S} \to V$ is a smooth map of relative dimension e and that fiber by fiber $\mathcal{S} \to \mathcal{S}'$ is a resolution (in particular, the fibers of $\mathcal{S}$ and $\mathcal{S}'$ over V are birational).

Let $N = \dim(U)$. As the fibers of $\mathcal{S}$ and $\mathcal{S}'$ are birational, if $S_0 \subset \mathcal{S}$ is any fiber, then the exterior power of the differential:

$$\wedge^{e+N} df \colon f^* \left(\wedge^{e+N} \Omega_{\mathcal{X}} \right) |_{S_0} \to \omega_{\mathcal{S}}|_{S_0}$$

is not identically 0. The normal bundle of $S_0 \subset \mathcal{S}$ is trivial and hence

$$\sigma : \omega_{\mathcal{S}}|_{S_0} \cong \omega_{S_0}.$$

Moreover, the exterior product gives rise to the isomorphism:

$$f^* \left(\wedge^{e+N} \Omega_{\mathcal{X}} \right) \Big|_{S_0} \cong f^* \left(\wedge^{n-e} T_{\mathcal{X}}(1) \right) \otimes f^*(\omega_{\mathcal{X}}(e-n))|_{S_0}.$$

The triviality of the normal bundle of $X \subset \mathcal{X}$ implies $\omega_{\mathcal{X}}|_X = \omega_X$. So if we twist the map $\wedge^{e+N} df$ by $(\omega_X(e-n))^{-1}$, use the isomorphism σ, and the assumption that $T_{\mathcal{X}}(1)|_X$ is globally generated, we see that the line bundle $\mathcal{O}_{S_0}(E) := \omega_{S_0} \otimes f^*(\omega_X(e-n))^{-1}|_{S_0}$ is effective. Therefore we have

$$\omega_{S_0} = f^*(\omega_X(e-n))|_{S_0} \otimes \mathcal{O}_{S_0}(E)$$

is the tensor product of a line bundle which satisfies $(\mathrm{BVA})_{p+e}$ and an effective line bundle. Thus ω_{S_0} satisfies $(\mathrm{BVA})_{p+e}$ and we are done by [4, Thm. 1.10]. $\square$

So we are interested in showing that $T_{\mathcal{X}}(1)|_X$ is globally generated. Recall that the *kernel bundle*, $\mathcal{M}_L$, associated to a very ample line bundle L on $\mathbf{Y}$ is the kernel of the evaluation map:

$$\mathcal{M}_L := \ker(H^0(\mathbf{Y}, L) \xrightarrow{\text{eval}} L).$$

We have the following lemma.

Lemma 2.9. *Let $X \in |L|$ be a smooth divisor. Assume that*

$$H^1(X, \mathcal{M}_L(1)|_X) = 0$$

and that $\mathcal{M}_L(1)|_X$ is globally generated. Then $T_{\mathcal{X}}(1)|_X$ is globally generated.

Proof. To start we note that there is a map between the normal sequence of X in $\mathcal{X}$ and the normal sequence of X in $\mathbf{Y}$

$$\begin{tikzcd}
0 \arrow[r] & T_X(1) \arrow[r] \arrow[d, equal] & T_{\mathcal{X}}(1)|_X \arrow[r] \arrow[d, "dp_2"] & H^0(\mathbf{Y}, L) \otimes_{\mathbb{C}} \mathcal{O}_X(1) \arrow[r] \arrow[d, "\text{eval}"] & 0 \\
0 \arrow[r] & T_X(1) \arrow[r] & T_{\mathbf{Y}}(1)|_X \arrow[r] & L(1)|_X \arrow[r] & 0
\end{tikzcd}$$

Then by the snake lemma, we have

$$\mathcal{M}_L(1)|_X = \ker(dp_2) = \ker(\text{eval}).$$

To show $T_{\mathcal{X}}(1)|_X$ is globally generated consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{M}_L(1)|_X) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d, "\text{eval}"] & H^0(T_{\mathcal{X}}(1)|_X) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d, "\text{eval}"] & H^0(T_{\mathbf{Y}}(1)|_X) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d, "\text{eval}"] & 0 \\
0 \arrow[r] & \mathcal{M}_L(1)|_X \arrow[r] \arrow[d] & T_{\mathcal{X}}(1)|_X \arrow[r, "dp_2"] & T_{\mathbf{Y}}(1)|_X \arrow[r] \arrow[d] & 0. \\
 & 0 & & 0 &
\end{tikzcd}$$

The left and right evaluation maps are surjective - the left by the assumption that $\mathcal{M}_L(1)|_X$ is globally generated and the right because $\mathbf{Y}$ is a rational homogeneous space so $T_{\mathbf{Y}}$ is globally generated. Then by the snake lemma, the center evaluation map is surjective and thus $T_{\mathcal{X}}(1)$ is globally generated. $\square$

Finally, if $L = \mathcal{O}_{\mathbf{Y}}(d)$ as in the case of quadrics, then $X \subset \mathbf{P}^M$ is projectively normal. So we can apply the following result due to Ein.

Proposition 2.10 (See [6, Prop. 1.2(c)]). *If $L = \mathcal{O}_{\mathbf{Y}}(d)$ is a multiple of $\mathcal{O}_{\mathbf{Y}}(1)$, then $\mathcal{M}_L(1)|_X$ is globally generated.*

Now we can easily prove Lemma 2.6.

Proof of Lemma 2.6. As $\omega_X = \mathcal{O}_X(d-n-1)$ we have we have that $\omega_X(-n)$ is p-very ample for $p = d-2n-1$. If we know that $H^1(X, \mathcal{M}_L(1)|_X) = 0$ by Propositions 2.8, Lemma 2.9, and Proposition 2.10 we obtain the inequality:

$$c \geq e + d - 2n + 1.$$

So it remains to show $H^1(X, \mathcal{M}_L(1)|_X) = 0$. Twisting the exact sequence that defines $\mathcal{M}_L$ by $\mathcal{O}_{\mathbf{Q}}(1)$ and restricting to X gives the following long exact sequence on cohomology.

$$\begin{aligned} \cdots \to H^0(\mathbf{Q}, \mathcal{O}_{\mathbf{Q}}(d)) \otimes H^0(X, \mathcal{O}_X(1)) &\xrightarrow{\text{eval}} H^0(X, \mathcal{O}_X(d+1)) \\ \to H^1(X, \mathcal{M}_L(1)|_X) \to H^0(\mathbf{Q}, \mathcal{O}_{\mathbf{Q}}(d)) &\otimes H^1(X, \mathcal{O}_X(1)) \to \cdots \end{aligned}$$

Now the evaluation map is surjective because $X \subset \mathbf{P}^{n+2}$ is projectively normal, and $H^1(X, \mathcal{O}_X(1)) = 0$ is an easy computation. Therefore, $H^1(X, \mathcal{M}_L(1)|_X)$ vanishes. □

3. Cubics

Let $X_d \subset Z \subset \mathbb{P}^{n+2}$ be a complete intersection of type $(3, d)$ in a cubic hypersurface Z. In this section, we prove Theorem B, which calculates the degree of irrationality of X_d for $n = 2, 3$. Our proof depends on Theorem 1.9, which describes the geometry of the fibers of low degree rational maps $X_d \dashrightarrow \mathbb{P}^n$, as well as known theorems about the geometry of Fano varieties of cubic threefolds and cubic fourfolds.

First we give upper and lower bounds on the degree of irrationality of X.

Lemma 3.1. *Let $X = X_d \subset Z \subset \mathbf{P}^{n+2}$ be a smooth divisor in a smooth cubic hypersurface with $X \in |\mathcal{O}_Z(d)|$. If $n = 2$ or 3 and $d \geq 5n-2$ then there are no maps:*

$$\phi\colon X \dashrightarrow \mathbf{P}^n$$

with fibers lying on lines in $\mathbf{P}^{n+2}$. As a consequence

$$2(d-n) + 2 \leq \operatorname{irr}(X) \leq 2d.$$

Proof. For the upper bound, we can choose a line contained in Z that meets X in a zero-dimensional subscheme of length d. Projection from such a line yields a rational map of degree $2d$.

For the lower bound, consider a dominant rational map $\phi\colon X \dashrightarrow \mathbf{P}^n$. For the sake of contradiction, assume $\deg(\phi) \leq 2(d-n)+1$. Since $\omega_X = \mathcal{O}_X(d-n)$, by Proposition 1.7 a general fiber ξ of ϕ satisfies $\mathrm{CB}(d-n)$. By Lemma 1.8, ξ lies on a line ℓ. Moreover,

$$\#\xi = \deg(\phi) \geq d-n+2 > 3,$$

which implies that ℓ must be contained in Z. Thus, as in the proof of Lemma 2.5, we obtain a rational map

$$\mathbf{P}^n \dashrightarrow \mathrm{Fano}(Z).$$

If $n=2$, $\mathrm{Fano}(Z)$ is the so-called *Fano surface*, which embeds into its Albanese (see [5]). The Albanese is an abelian fivefold and thus contains no rational curves. Thus, any such rational map is constant. This implies that a general point on the surface X lies on a single line, a contradiction.

If $n=3$, then the Fano variety is a hyperkähler manifold of dimension 4 (see [3]). The smooth locus of the image of $\mathbf{P}^n$ in $\mathrm{Fano}(Z)$ must be Lagrangian with respect to the symplectic form. So the image has dimension ≤ 2. If this was possible, then X would be covered by lines. This is a contradiction as X is of general type. Thus, $\deg(\phi) \geq 2(d-n)+2$, as desired. □

Proof of Theorem B. Let $\phi : X \dashrightarrow \mathbf{P}^n$ be a map of minimum degree $\delta = \mathrm{irr}(X)$. By Lemma 3.1, if $d \geq 5n-2$ then

$$\deg(\phi) \leq 2d \leq \frac{5}{2}(d-n)+1,$$

and the fibers of ϕ are not contained in lines in $\mathbf{P}^{n+2}$. By Theorem 1.9, a general fiber ξ is contained in a curve C of degree 2. If $C = \ell_1 \cup \ell_2$ is a union of two lines, then by Remark 1.11 each line contains at least $d-n+1 \geq 4$ points. Thus both lines are contained in Z. Likewise, if C is a smooth plane conic, then $C \cap Z$ contains at least $\#\xi \geq 2d > 6$ points. So again, $C \subset Z$.

First assume $n=2$, and that a general fiber ξ is contained in the union of two lines $C = \ell_1 \cup \ell_2 \subset Z$. This gives a rational map

$$\mathbf{P}^2 \dashrightarrow \mathrm{Sym}^2(\mathrm{Fano}(Z)).$$

Since $\mathrm{Fano}(Z)$ embeds into its Albanese, the above rational map yields the following commutative diagram:

$$\begin{tikzcd}
\mathbf{P}^2 \arrow[d, dashed] \arrow[rrd, dashed, bend left] & & \\
\mathrm{Sym}^2(\mathrm{Fano}(Z)) \arrow[r, hook] & \mathrm{Sym}^2(\mathrm{Alb}(\mathrm{Fano}(Z))) \arrow[r, "\Sigma"'] & \mathrm{Alb}(\mathrm{Fano}(Z)).
\end{tikzcd} \tag{4}$$

Here Σ is the map which adds the degree 2 cycles in the group law of $\mathrm{Alb}(\mathrm{Fano}(Z))$. As $\mathbf{P}^2$ is rationally connected, the image of $\mathbf{P}^2$ in $\mathrm{Sym}^2(\mathrm{Alb}(\mathrm{Fano}(Z)))$ must be contained in a fiber of Σ. A fiber of Σ is the Kummer variety K of $\mathrm{Alb}(\mathrm{Fano}(Z))$. By [11, Thm. 1], the rational curves on K are rigid, so the closure of the image of $\mathbf{P}^2$ in K is either a point or a rational curve. Both cases are impossible, the first for dimension reasons. The second case would imply that X is contained in a rational surface, which is impossible as X is a surface of general type.

Therefore, a general fiber ξ of ϕ is contained in a smooth plane conic $C \subset Z$. If $\pi = \pi_\xi$ is the plane spanned by C, then $\pi \cap Z = C \cup \ell_\xi$, where ℓ_ξ is the residual line to C contained in Z. Again this determines a rational map

$$\mathbf{P}^2 \dashrightarrow \mathrm{Fano}(Z) \text{ by } y \mapsto [\ell_\xi] \text{ (where } \xi = \phi^{-1}(y)).$$

As above, this map must be constant. So all the conics are residual to the same line $\ell_\xi \subset Z$. Thus the map ϕ is given by projection from this line up to postcomposition with a Cremona transformation, and

$$\mathrm{irr}(X) = \begin{cases} 2d - 2 \text{ if } X \text{ contains a line,} \\ 2d \text{ otherwise.} \end{cases}$$

Now assume $n = 3$, and for contradiction assume $\delta < 2d$. Let $\xi = \phi^{-1}(y)$ be a general fiber which is contained in a degree 2 curve $C \subset Z$. As ξ is general, no component of C is contained in X (because X is of general type). Thus the intersection $C \cap X$ is a 0-dimensional scheme of length $2d$, of which δ points are accounted for. For a general point $y \in \mathbf{P}^3$, we can associate to y the *residual* effective 0-cycle $\zeta_y := [C \cap X] - [\xi]$ which has degree $e = 2d - \delta$. By Lemma 3.1, $e \leq 4$.

We claim that the cycle ζ_y is not a constant cycle. First, note that the degree 2 curves $C \subset Z$ must sweep out all of Z, because they sweep out some uniruled subvariety of Z which contains the general type threefold X.

Consider the case when $C = \ell_1 \cup \ell_2$ is the union of two lines. We claim that each ℓ_i meets ξ at the same number of points. Assuming this, if ζ_y is constant it has points on both ℓ_1 and ℓ_2. As Z is not a cone, for any fixed point $z \in Z$ (and thus for any finite set of points) a general point of Z cannot be connected to z via a line $\ell \subset Z$. But Z is swept out by the lines ℓ_i which meet ζ_y, thus the ζ_y cannot be a constant cycle.

To complete the argument in the previous paragraph we must show that each ℓ_i meets ξ at the same number of points. As $y \in \mathbf{P}^3$ is general, there is an open subset $y \in U \subset \mathbf{P}^3$ such that the map:

$$\phi|_{X_U} : X_U := \phi^{-1}(U) \to U$$

is a topological covering map. After further shrinking U we can guarantee that $\ell_1 \cap \ell_2 \cap \xi = \emptyset$ (if this were not the case then the intersection $\ell_1 \cap \ell_2$ would define a rational section of ϕ over U which is absurd). Then we can factor the map through a space

$$W := \left\{ (y, [\ell]) \in U \times \mathrm{Fano}(Z) \,\middle|\, \ell \text{ is a line meeting } \xi_y \text{ at more than 3 points.} \right\}$$

It follows that $W \to U$ is a 2-to-1 topological covering and that the map

$$\psi : X_U \to W$$

defined by mapping a point in x to the line ℓ_i containing x is also a topological covering. Therefore every fiber of ϕ has the same number of points, which implies that each ℓ_i meets ξ at the same number of points.

In the case C is a smooth plane conic, suppose for contradiction that there is a point $z \in X$ which is contained in ζ_y for all general $y \in \mathbf{P}^3$. Let P be the plane spanned by C. As in the $n = 2$ case, the conic determines a residual line defined by $\ell_y \cup C = P \cap Z \subset Z$. If $z \in \ell_y$ for general $y \in \mathbf{P}^3$ then the plane spanned by C is contained in the tangent hyperplane to Z at z. This means the conics in the family do not sweep out all of Z, a contradiction. So we can assume that for a general point $y \in \mathbf{P}^3$, the residual line ℓ_y does not contain z. But then the point $z \in X$ and the line ℓ_y span the plane P, and thus the conic C is determined by ℓ_y and z. This implies that the rational map

$$\mathbf{P}^3 \dashrightarrow \mathrm{Fano}(Z) \text{ which sends } y \mapsto [\ell_y]$$

is birational onto its image. This is a contradiction as the image must be Lagrangian (see the proof of Lemma 3.1). Therefore, the cycle ζ_y is not constant.

Let $\lambda \subset \mathbf{P}^3$ be a line through a general point in $\mathbf{P}^3$ such that the closure of

$$\bigcup_{y \in \lambda} \zeta_y \subset X$$

is positive dimensional. Define the incidence correspondence:

$$D = \overline{\{(a, y) \in X \times \mathbf{P}^3 | a \in \zeta_y, y \in \lambda\}} \subset X \times \mathbf{P}^3.$$

Then there is a 1-dimensional component $D_0 \subset D$ such that neither of the projections $D_0 \to X$ or $D_0 \to \lambda$ are constant. The projection to λ shows that $\mathrm{gon}(D_0) \leq \deg(\zeta_y) \leq 4$. Thus the image of D_0 in X is a curve E with gonality ≤ 4. As we assumed X is very general (in particular it is contained in a very general hypersurface in $\mathbf{P}^5$) by [4, Propn. 3.8],

$$\mathrm{gon}(E) \geq d - 2 \cdot 4 + 1 = d - 7 \geq 6,$$

which is a contradiction. □

4. Complete intersections of two quadrics

Let $Z = Q_1 \cap Q_2 \subset \mathbf{P}^5$ be a smooth intersection of two quadrics and let

$$X = X_d \subset Z$$

be a smooth surface in the linear series $X \in |\mathcal{O}_Z(d)|$. The goal of this section is to prove Theorem C. That is if $d \geq 8$ then

$$\mathrm{irr}(X) = \begin{cases} 2d - 2 \text{ if } X \text{ contains a plane conic,} \\ 2d - 1 \text{ if } X \text{ contains a line and no conics,} \\ 2d \text{ otherwise.} \end{cases}$$

Moreover, we will prove that any map:

$$\phi\colon X \dashrightarrow \mathbf{P}^2$$

of degree $\leq 2d$ is given by projection from a plane in $\mathbf{P}^5$.

To start, we recall some classical results about the projective geometry of a smooth (2,2) complete intersection Fano threefold. For every such threefold Z there is an associated genus 2 hyperelliptic curve, C_Z which can be defined by an equation given as follows. Let M_1 and M_2 be the symmetric matrices corresponding to the quadratic forms determined by Q_1 and Q_2 respectively. Then C_Z is the hyperelliptic curve defined as the compactification of the affine curve:

$$\left(y^2 = \det(M_1 + tM_2)\right) \subset \mathbb{C}^2.$$

In particular, the branch points of the hyperelliptic map

$$h_Z\colon C_Z \xrightarrow{2:1} \mathbf{P}^1$$

correspond to singular quadrics $Q_t \in |H^0(I_Z(2))| \cong \mathbf{P}^1$.

Remark 4.1. The assumption that Z is smooth implies that for all $Q_t \in |I_Z(2)|$, Q_t has at worst isolated singularities. I.e. for all t the matrix

$$M_t = M_1 + tM_2$$

has rank ≥ 5. Moreover, smoothness of Z implies C_Z is smooth.

Another way to define C_Z is to look at the incidence variety

$$\mathrm{Inc}_Z = \left\{(P,t)\,\middle|\, P \subset Q_t \text{ is a 2-plane in the quadric } Q_t \in |H^0(I_Z(2))|\right\} \subset \mathrm{Gr}(3,6) \times \mathbf{P}^1.$$

Then C_Z can be defined as the Stein factorization of the projection to $t \in \mathbf{P}^1$:

$$\begin{tikzcd} & C_Z \arrow[dr, "h_Z"] & \\ \mathrm{Inc}_Z \arrow[ur] \arrow[rr] & & \mathbf{P}^1. \end{tikzcd}$$

Remark 4.2. The fiber of $\mathrm{Inc}_Z \to \mathbf{P}^1$ over $t \in \mathbf{P}^1$ is the Fano variety of planes in Q_t, denoted Fano(2, Q_t). There are two possibilities for Fano(2, Q_t):

(1) Q_t is smooth, and Fano(2, Q_t) $\cong \mathbf{P}^3 \sqcup \mathbf{P}^3$, or
(2) Q_t has an isolated singularity, and Fano(2, Q_t) $\cong \mathbf{P}^3$.

Historically, people have been interested in relating various aspects of the projective geometry of Z to the geometry of the curve C_Z. For our purposes the most important result is the following theorem due to Narasimhan and Ramanan.

Theorem 4.3 ([10, Thm. 5]). *Let $Z \subset \mathbf{P}^5$ be a smooth complete intersection of two quadrics. Let* $\mathrm{Fano}(Z)$ *be the associated Fano variety of lines in Z. Then*

$$\mathrm{Fano}(Z) \cong \mathrm{Jac}(C_Z).$$

Now we proceed to prove Theorem C.

Proposition 4.4. *Let $X \in |\mathcal{O}_Z(d)|$ be a smooth hypersurface. There exists a rational map:*

$$\phi_0 \colon X \dashrightarrow \mathbf{P}^2$$

such that $\deg(\phi_0) = 2d$. *In particular,* $\mathrm{irr}(X) \leq 2d$.

Proof. Let $Q_0 \in |I_Z(2)|$ be a quadric in the ideal of Z and let $P \subset Q_0$ be a general plane in Q_t. Linear projection from P gives a rational map: $\pi_P \colon \mathbf{P}^5 \dashrightarrow \mathbf{P}^2$. Setting $\phi_0 = \pi_P|_X$, then ϕ_0 is dominant and $\deg(\phi_0) = 4d - \#(P \cap X) = 2d$. □

Remark 4.5. Let X be as above, and $P \subset Q_t$ a plane in a quadric $Q_t \in |I_Z(2)|$. Let $\phi_0 = \pi_P|_X$ be the restriction to X of the linear projection from P. There are three possibilites for $\deg(\phi_0)$:

(1) $\deg(\phi_0) = 2d$ if the intersection $P \cap X$ is 0-dimensional,
(2) $\deg(\phi_0) = 2d - 1$ if the intersection $P \cap X$ has a single 1-dimensional component which is a line, or
(3) $\deg(\phi_0) = 2d - 2$ if the intersection $P \cap X$ contains a plane conic.

For example, to show (2), if X contains a line $\ell \subset X$ and we project from a plane $P \supset \ell$ which contains ℓ then the fibers of the rational map $\pi_P \colon \mathbf{P}^5 \dashrightarrow \mathbf{P}^2$ are 3-planes $\mathbf{P}^3 \supset P$ containing P (or more precisely, the complement $\mathbf{P}^3 \setminus P$). So we have

$$\deg(\phi_0) = \#\{x \in X \cap \mathbf{P}^3 | x \notin P\}.$$

Suppose $\mathbf{P}^3$ is a general 3-plane containing P. Then $X \cap \mathbf{P}^3$ is the intersection of two quadrics and a degree d hypersurface in $\mathbf{P}^3$. We have $Q_0 \cap \mathbf{P}^3 = P \cup P'$ is the union of two planes. Moreover, ℓ is contained in both quadrics and the degree d hypersurface.

As $\mathbf{P}^3$ is general, $P \cap P' = \ell'$ is a line and $\ell \cap \ell' = p$ is a single point. Thus

$$\deg(\phi_0) = \#\{x \in X \cap P' | x \notin \ell'\}.$$

Finally, $X \cap P'$ is the complete intersection of a conic C_1 and a degree d curve C_2. By our genarality assumption, the only point in $C_1 \cap C_2 \cap \ell'$ is p. Thus $\deg(\phi_0) = 2d - 1$. A similar analysis gives (3).

Now we would like to prove that if

$$\phi \colon X \dashrightarrow \mathbf{P}^2$$

is a rational map with $\deg(\phi) \leq 2d$, then ϕ is given by projection from a plane $P \subset Q_t$ for some $Q_t \in |I_Z(2)|$. We start by applying Theorem 1.9.

Lemma 4.6. *Let*

$$\phi \colon X \dashrightarrow \mathbf{P}^2$$

be a dominant rational map with $\delta = \deg(\phi) \leq 2d$. Then $\delta \geq 2d - 2$. Moreover, if $\xi = \phi^{-1}(p)$ is a general fiber of ϕ, then ξ is contained in a smooth conic $C \subset Z$.

Proof. By adjunction, the canonical bundle of X is $\omega_X = \mathcal{O}_X(d-2)$. Thus ξ satisfies Cayley–Bacharach with respect to the linear series $|\mathcal{O}_{\mathbf{P}^5}(d-2)|$. The assumption that $d \geq 8$ implies that $\delta \leq 5/2(d-2)+1$. Thus by Theorem 1.9 we know that one of the following holds:

(1) ξ is contained in a line $\ell \subset \mathbf{P}^5$,
(2) ξ is contained in a union of two lines $\ell_1 \cup \ell_2 \subset \mathbf{P}^5$, or
(3) ξ is contained in a smooth plane conic $C \subset \mathbf{P}^5$.

We now prove that the first 2 cases are impossible. We follow the same argument as in the proof of Theorem B(1).

Assume for contradiction that we are in case (1), i.e. $\xi \subset \ell$. Then as $d \geq 8$ we know that $\delta \geq 6$ by Theorem 1.5. By Bezout's theorem $\ell \subset Z$. Thus, a general point in $\mathbf{P}^2$ parameterizes a line in Z, so we get a rational map:

$$\mathbf{P}^2 \dashrightarrow \mathrm{Fano}(Z) \cong \mathrm{Jac}(C_Z).$$

This map must be constant as $\mathrm{Jac}(C_Z)$ contains no rational curves. Therefore, we have that every general point in X is contained in a single line, a contradiction. As ξ is not contained in a line Lemma 1.8 implies that $\delta \geq 2d - 2$.

Now assume for contradiction that ξ is in the union of 2 distinct lines ℓ_1 and ℓ_2, i.e. assume we are in case (2). Then by Remark 1.11 we have that at least $d-1$ points lie on each line, thus $\ell_1, \ell_2 \subset Z$. Then a general point $p \in \mathbf{P}^2$ parameterizes a pair of lines $\ell_1 \cup \ell_2$, and we get a rational map:

$$\psi \colon \mathbf{P}^2 \dashrightarrow \mathrm{Sym}^2(\mathrm{Fano}(C_Z)) \cong \mathrm{Sym}^2(\mathrm{Jac}(C_Z)).$$

The image of such a map must lie in a single fiber of the addition map:

$$\Sigma \colon \mathrm{Sym}^2(\mathrm{Fano}(C_Z)) \rightarrow \mathrm{Jac}(C_Z).$$

The fibers of Σ are a singular Kummer K3 surfaces. In particular, the fibers are not uniruled. Thus the image of ψ has dimension at most 1. This implies X is contained in a ruled surface, which is a contradiction as X is a general type surface.

Finally, assume ξ is contained in a smooth conic $C \subset \mathbf{P}^5$. As $d \geq 8$, we have $\#(C \cap Z) \geq \delta \geq 8$. Bezout's theorem implies $C \subset Z$. □

Lemma 4.7. *Let Z be a smooth (2,2)-complete intersection in $\mathbf{P}^5$, let $C \subset Z$ a plane conic, and let P be the plane spanned by C. Then there is a unique quadric in the pencil*

$$Q_t \in |I_Z(2)|$$

such that $P \subset Q_t$.

Proof. First, it is clear that there is at most one such quadric, as a smooth (2,2)-complete intersection in $\mathbf{P}^5$ contains no planes. Now consider the restriction map:

$$r\colon H^0(\mathbf{P}^5, I_Z(2)) \to H^0(P, I_C(2)).$$

We have $H^0(\mathbf{P}^5, I_Z(2))$ is 2-dimensional and $H^0(P, I_C(2))$ is 1-dimensional. The map r is nonzero as $P \not\subset Z$. Thus r is surjective, and the kernel of r is 1-dimensional, spanned by the equation of Q_t. □

Now given a rational map $\phi\colon X \dashrightarrow \mathbf{P}^2$ with $\deg(\phi) \leq 2d$, Lemmas 4.6 and 4.7 imply that a general point $t \in \mathbf{P}^2$ parameterizes a plane P_t which is contained in a quadric $Q_t \in |I_Z(2)|$. All together, this gives a rational map:

$$\mathbf{P}^2 \dashrightarrow \mathrm{Inc}_Z.$$

Note that as C_Z is a smooth genus 2 curve the composition:

$$\mathbf{P}^2 \dashrightarrow \mathrm{Inc}_Z \to C_Z$$

must be constant. Thus there is some fixed quadric $Q_t \in |I_Z(2)|$ such that the above rational map factors as

$$\begin{tikzcd} \mathbf{P}^2 \arrow[rr, dashed] \arrow[dr, dashed] & & \mathrm{Inc}_Z. \\ & \mathrm{Fano}(2, Q_t) \arrow[ur] & \end{tikzcd}$$

By Remark 4.2, there are two possibilities for $\mathrm{Fano}(2, Q_t)$. Either

(1) Q_t is smooth, and $\mathrm{Fano}(2, Q_t) \cong \mathbf{P}^3 \sqcup \mathbf{P}^3$, or
(2) Q_t has an isolated singularity, and $\mathrm{Fano}(Q_t) \cong \mathbf{P}^3$.

In either case the rational map $\mathbf{P}^2 \dashrightarrow \mathrm{Fano}(2, Q_t)$ lands in a single $\mathbf{P}^3$. Let B be the closure of the image of $\mathbf{P}^2$ in $\mathbf{P}^3$, and consider the following diagram:

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d] & G \arrow[r, "\pi"] \arrow[d, "\psi"] & Q_t. \\ B \arrow[r, hook] & \mathbf{P}^3 & \end{tikzcd} \tag{5}$$

Here G is the universal plane over $\mathbf{P}^3$, and F the family of planes over B, i.e. $F = B \times_{\mathbf{P}^3} G$.

Lemma 4.8. *Let*

$$f\colon \mathbf{P}^2 \dashrightarrow B \subset \mathbf{P}^3$$

be the map induced by $\phi\colon X \dashrightarrow \mathbf{P}^2$*. Then* f *is birational.*

Proof. Start by resolving the indeterminacy of f:

$$\begin{tikzcd} B' \arrow[d] \arrow[dr, "f'"] & \\ \mathbf{P}^2 \arrow[r, dashed, "f"'] & B. \end{tikzcd} \tag{6}$$

It suffices to show that f' is birational. First we prove that B is a surface, i.e. that f' is generically finite. Note that

$$\pi^{-1} Z \cap F \to B$$

is the family of conics in Z parameterized by B, and has dimension $\dim(B) + 1$. A general point of X is contained in a conic in this family. Thus, $\pi(\pi^{-1}(Z) \cap F)$ is a subvariety of Z containing the divisor X. Moreover X cannot be a component of $\pi(\pi^{-1}(Z) \cap F)$ as X is not uniruled. Therefore, $\pi(\pi^{-1}(Z) \cap F) = Z$ which by a dimension count shows $\dim(B) \geq 2$.

Now assume for contradiction that $\deg(f') \geq 2$. For every general point $x \in B$, let P_x be the plane in Q_t which is parameterized by the point x and let $C_x = P_x \cap Z$ be the smooth conic in Z parameterized by x. Note that as x is general, C_x is not contained in X as X is not uniruled. Thus the interesection $C_x \cap X$ is proper. If $\deg(f') \geq 2$ then there are at least two fibers of ϕ which are contained in $C_x \cap X$. Then we have

$$2d = \text{length}(C_x \cap X) \geq \#(C_x \cap X) \geq 2\delta \geq 2(2d - 2).$$

This contradicts the assumption that $d \geq 8$. □

Lemma 4.9. *If $B \subset \mathbf{P}^3$ has degree 1 (i.e. B is a plane) then the congruence B corresponds to the closure of the fibers of a projection from a plane, and thus ϕ is birationally equivalent to projection from a plane in Q_t.*

Proof. It is straightforward to show that the fibers of projection from a plane in Q_t give rise to a plane $B \subset \mathbf{P}^3$. A parameter count shows that all planes in $\mathbf{P}^3$ arise this way. □

Proof of Theorem C. By Lemma 4.9, what remains to show is that for any map:

$$\phi \colon X \dashrightarrow \mathbf{P}^2$$

the corresponding surface $B \subset \mathbf{P}^3$ is a plane. First, if $x \in Q_t$ is a smooth point, then the fiber

$$\pi^{-1}(x) \cong \mathbf{P}^1_x$$

maps isomorphically onto a line in $\mathbf{P}^3$. Thus if x is general, then

$$\deg(B \subset \mathbf{P}^3) = \#(\psi(\mathbf{P}^1_x) \cap B) = \#(\mathbf{P}^1_x \cap \psi^{-1}(B)) = \#(\mathbf{P}^1 \cap F) = \deg(\pi|_F).$$

Therefore we want to prove that $\delta = \deg(\pi|_F) = 1$. Note that the following holds

$$\pi_*(F \cdot \pi^{-1}(X)) = \deg(\pi|_F) \cdot [X],$$

so our strategy will be to understand the intersection $F \cdot \pi^{-1}(X)$ as a cycle.

First we claim that the intersection of these varieties is proper. As F is an irreducible divisor and $\pi^{-1}(X)$ is also irreducible, it suffices to show that $\pi^{-1}(X) \not\subset F$. This follows because the map:

$$\psi|_{\pi^{-1}(X)} : \pi^{-1}(X) \to \mathbf{P}^3$$

is surjective, but $\psi(F) = B \subsetneq \mathbf{P}^3$. Thus $F \cdot \pi^{-1}(X)$ is a positive linear combination of subvarieties supported on the intersection $F \cap \pi^{-1}(X)$.

Define a rational map

$$X \dashrightarrow Q_t \times \mathbf{P}^3$$

by sending a general point $x \in X$ to the pair $(x, \phi(x))$. Let X' denote the closure of the image of this map. Note that $G \subset Q_t \times \mathbf{P}^3$ and moreover $X' \subset F \cap \pi^{-1}(X) \subset G$. In particular, this implies X' is a component of $F \cap \pi^{-1}(X)$, and thus we have

$$F \cdot \pi^{-1}(X) = a[X'] + \sum b_i [E_i]$$

with $a, b_i \geq 0$. Now we can compute

$$\begin{aligned} 2d[B] &= \deg(\psi : \pi^{-1}(X) \to \mathbf{P}^3)[\mathbf{P}^3] \cdot B \\ &= \psi_*(\pi^{-1}(X)) \cdot B \\ &= \psi_*(\pi^{-1}(X) \cdot \psi^* B) \\ &= \psi_* \left(a[X'] + \sum b_i [E_i] \right) \\ &= a \deg(\psi|_{X'})[B] + \sum b_i \deg(\psi|_{E_i})[B] \\ &= \left(a\delta + \sum b_i \deg(\psi|_{E_i}) \right) [B]. \end{aligned}$$

I.e. we have

$$2d = a\delta + \sum b_i \deg(\psi|_{E_i}). \tag{7}$$

On the right hand side all the terms are positive, except for possibly the $\deg(\psi|_{E_i})$ which can be 0. As $\delta \geq 2d - 2$ we know that $a = 1$.

Now assume there is an E_i such that

$$\deg(\pi|_{E_i} : E_i \to X) \neq 0,$$

i.e. $\pi_*(E_i) \neq 0$. This actually implies that the map

$$\psi|_{E_i} : E_i \to B$$

is surjective (if this were not the case, the fibers of $\psi|_{E_i}$ would be plane conics, which would imply X is uniruled). Then E_i gives a correspondence between X

and B. Proposition 1.7 implies the general fibers of $\psi|_{E_i}$ satisfy CB$(d-2)$, so by Proposition 1.8 we have $\deg(\psi|_{E_i}) \geq d$. By the assumption $d \geq 8$, this contradicts (7). Therefore, there are no E_i such that $\pi_*(E_i) \neq 0$.

Thus, we have

$$\delta[X] = \pi_*(F \cdot \pi^{-1}(X)) = \pi_*([X']) + \sum b_i \pi_*([E_i]) = [X] + 0,$$

which proves $\delta = 1$. □

5. Grassmannians

Let $k \neq 1, m-1$, and let $\mathbf{G} = \mathrm{Gr}(k,m) \subset \mathbf{P}$ be the Plücker embedding of the Grassmannian of k-planes in an m-dimensional vector space. The aim of this section is to prove Theorem D, that is if

$$X = X_d \subset \mathbf{G}$$

is a very general hypersurface with $X \in |\mathcal{O}_{\mathbf{G}}(d)|$ and $d \geq 3m-5$ then $\mathrm{irr}(X) = d$.

To start we show $\mathrm{irr}(X) \leq d$.

Lemma 5.1. *Let $X \in |\mathcal{O}_{\mathbf{G}}(d)|$ and set $n := \dim(X) = \dim(\mathbf{G}) - 1$. There exists a degree d map*

$$\phi_0 \colon X \dashrightarrow \mathbf{P}^n.$$

Proof. To start we show there is a rational map

$$p \colon \mathbf{G} \dashrightarrow \mathbf{P}^n$$

such that every fiber of p is in a line $\ell \subset \mathbf{P}$ that is contained in $\mathbf{G}$. Choose a one dimensional subspace $\lambda \subset \mathbb{C}^m$, an $(m-1)$-dimensional subspace $W \subset \mathbb{C}^m$, and let

$$T : \mathbb{C}^m \to (\mathbb{C}^m/\lambda)$$

denote the quotient map. Let

$$\mathrm{Fl}(k-1,k,m-1) := \left\{ [U \subset V \subset (\mathbb{C}^m/\lambda)] \;\middle|\; \begin{array}{c} \text{Where } U \text{ and } V \text{ are subspaces of } (\mathbb{C}^m/\lambda) \\ \text{of dimensions } k-1 \text{ and } k \text{ respectively} \end{array} \right\}$$

denote the the partial flag variety of $(\mathbb{C}^m/\lambda)$. Then we can define a rational map from $\mathbf{G}$ to $\mathrm{Fl}(k-1,k,m-1)$ as follows:

$$\begin{aligned} p = p_{\lambda,W} \colon \mathbf{G} &\dashrightarrow \mathrm{Fl}(k-1,k,m-1). \\ [\Lambda \subset \mathbb{C}^m] &\mapsto [T(\Lambda \cap W) \subset T(\Lambda) \subset (\mathbb{C}^m/\lambda)] \end{aligned}$$

Note that $\mathrm{Fl}(k-1,k,m-1) \simeq_{\mathrm{bir}} \mathbf{P}^n$, and two general points $[\Lambda \subset \mathbb{C}^m]$ and $[\Lambda' \subset \mathbb{C}^m]$ are in the same fiber of p if and only if Λ' satisfies $\Lambda \cap W \subset \Lambda' \subset \Lambda + \lambda$.

It is straightforward to show that the closure of all such $[\Lambda' \subset \mathbb{C}^m]$ form a line in the Plücker embedding of $\mathbf{G}$. Now set ϕ_0 equal to the composition

$$\phi_0 = p|_X : X \dashrightarrow \mathrm{Fl}(k-1, k, m-1) \simeq_{\mathrm{bir}} \mathbf{P}^n$$

By the construction of ϕ_0, the fiber of ϕ_0 over a general point in $\mathbf{P}^n$ is contained in a line $\ell \subset \mathbf{G} \subset \mathbf{P}$. An appropriate choice of λ and W will guarantee that $\ell \cap X$ does not meet the base locus of ϕ_0. Thus we have $\deg(\phi_0) = [\ell] \cdot [X] = d$. □

Now assume for contradiction that there is a dominant rational map

$$\phi : X \dashrightarrow \mathbf{P}^n$$

with $\deg(\phi) \le d - 1$. First, we show that all fibers of ϕ must lie on lines contained inside $\mathbf{G}$.

Lemma 5.2. *If $d \ge 2m - 2$ then a general fiber of ϕ lies on a line $\ell \subset \mathbf{P}$ which is contained in* $\mathbf{G}$.

Proof. The proof is the same as the proof of Lemma 2.4. We just remark that $\omega_X = \mathcal{O}_X(d - m)$. So by Theorem 1.5, $\deg(\phi) \ge d - m + 2$ and by Proposition 1.8 every fiber of ϕ lies on a line $\ell \subset \mathbf{P}$. The Grassmannian is cut out by quadrics, so by applying Bezout's theorem we have that $\ell \subset \mathbf{G}$. □

Thus a general point in $\mathbf{P}^n$ parameterizes a line in $\mathbf{G}$. This gives rise to a rational map from $\mathbf{P}^n$ to the Fano variety of lines in $\mathbf{G}$, which is $\mathrm{Fl}(k-1, k+1, m)$. As in §2 we get the following diagram:

$$\begin{tikzcd}
X & X' \arrow[l] \arrow[d, hook] & \\
\mathbf{G} \arrow[u, phantom] & F \arrow[l, "\pi"'] \arrow[d, "\psi"] & \\
& B \arrow[r, "f"] & \mathrm{Fl}(k-1, k+1, m).
\end{tikzcd} \tag{8}$$

Here $f : B \to \mathrm{Fl}(k-1, k+1, m)$ is a resolution of the indeterminacy of the map $\mathbf{P}^n \dashrightarrow \mathrm{Fl}(k-1, k+1, m)$. The variety F is the corresponding family of lines in $\mathbf{G}$ over B (with it's natural projections). Finally, X' is the closure of the image of the rational section $X \dashrightarrow B$ which sends a point x to $(x, \phi(x))$.

Lemma 5.3. *The map π is birational, i.e. the map ϕ determines a "congruence of lines of order one" on* $\mathbf{G}$.

Proof. The proof is identical to the proof of Lemma 2.5. □

We will also need the following lemma.

Lemma 5.4. *If $X \in |\mathcal{O}_{\mathbf{G}}(d)|$ is very general, then for any subvariety of X with dimension e and covering gonality c we have the inequality:*

$$c \ge e + d - m - n + 2.$$

Proof. As $\omega_X = \mathcal{O}_X(d-m)$ we have we have that $\omega_X(-n)$ is p-very ample for $p = d - m - n$. If we assume that $H^0(X, \mathcal{M}_L(1)|_X) = 0$ by Propositions 2.8, Lemma 2.9, and Proposition 2.10 we obtain the inequality:

$$c \geq e + d - m - n + 2.$$

For the proof that $H^0(X, \mathcal{M}_L(1)|_X)$ vanishes, see the proof of Lemma 2.6. $\square$

Proof of Theorem D. Assume for contradiction that there is a map

$$\phi\colon X \dashrightarrow \mathbf{P}^n,$$

with $\deg(\phi) = \delta \leq d - 1$. Then we can associate to ϕ the fundamental diagram (8). Let

$$\pi^*(X) = X' + \sum a_i E_i$$

where the E_i are irreducible exceptional divisors of the map $\pi : F \to \mathbf{G}$ and $a_i > 0$. Let ℓ be a fiber of ψ. As $X \in |\mathcal{O}_{\mathbf{G}}(d)|$ we have

$$d = [X] \cdot \pi_*[\ell] = \pi^*[X] \cdot [\ell] = \delta + \sum a_i [E_i] \cdot [\ell] = \delta + \sum a_i \deg(\psi|_{E_i}).$$

By the assumption that $\delta \leq d - 1$, there must by some $E = E_i$ such that $\deg(\psi|_E) \geq 1$. Set

$$c = \deg(\psi|_E).$$

Using that $\omega_X = \mathcal{O}_X(d-m)$, by Theorem 1.5 $\delta \geq d - m + 2$, which implies

$$1 \leq c \leq m - 2.$$

Let $e = \dim(\pi(E))$. As π is birational and $c \geq 1$, every point in $\mathbf{G}$ lies on a line which intersects the image of $\pi(E)$. If $[\Lambda] \in \mathbf{G}$, then the lines $\ell \subset \mathbf{G}$ through $[\Lambda]$ correspond to 2 step flags:

$$[U \subset V \subset \mathbb{C}^m] \in \mathrm{Fl}(k-1, k+1, m)$$

such that $U \subset \Lambda \subset V$. Every such flag is determined by the point $[U] \in \mathbf{P}(\Lambda)^\vee$ and the point $[V/\Lambda] \in \mathbf{P}(\mathbb{C}^m/\Lambda)$. Thus the union of all lines in $\mathbf{G}$ through $[\Lambda]$ has dimension $\leq \dim(\mathbf{P}(\Lambda)^\vee) + \dim(\mathbf{P}(\mathbb{C}^m/\Lambda) + 1 = m - 1$. This gives the estimate

$$e + m - 1 \geq n + 1.$$

Therefore the image $\pi(E)$ is a subvariety of X which has covering gonality $c \leq m - 2$ and dimension $e \geq n - m + 2$.

Plugging in our estimates for c and e into the inequality in Lemma 5.4, and using the assumption $d \geq 3m - 5$ we obtain the inequality:

$$\begin{aligned} m - 2 \geq c \geq e + d - m - n + 2 &\geq (n - m + 2) \\ &+ (3m - 5) - m - n + 2 = m - 1, \end{aligned}$$

which is a contradiction. $\square$

6. Products of projective space

Let $\mathbf{P} = \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_k}$ be a product of $k \geq 2$ projective spaces, and let

$$X = X_{(d_1,\ldots,d_k)} \subset \mathbf{P}$$

be a very general hypersurface with $X \in |\mathcal{O}_{\mathbf{P}}(d_1, \ldots, d_k)|$. The goal of this section is to prove Theorem E, i.e. if

$$\min\{d_i - m_i - 1\} \geq \max\{m_i\}$$

then $\mathrm{irr}(X) = \min\{d_i\}$. Throughout this section we define the following constants:

- $d := \min\{d_i\}$,
- $p := \min\{d_i - m_i - 1\}$,
- $m := \max\{m_i\}$, and
- $n := \dim(X) = m_1 + \cdots + m_k - 1$.

Thus the goal is to prove that if $p \geq m$ then $\mathrm{irr}(X) = d$.

To start we show that $\mathrm{irr}(X) \leq d$.

Lemma 6.1. *There is a degree d rational map*

$$\phi_0 \colon X \dashrightarrow \mathbf{P}^n.$$

Proof. It suffices to find a rational map of degree d to any n-dimensional rational variety. Without loss of generality assume that $d_1 = d$. Let $x \in \mathbf{P}^{m_1}$ be a general point in the first projective space. Consider the linear projection from x:

$$\pi_x \colon \mathbf{P}^{m_1} \dashrightarrow \mathbf{P}^{m_1 - 1}.$$

Let pr_i denote the ith projection $\mathrm{pr}_i \colon \mathbf{P} \to \mathbf{P}^{m_i}$, and consider the rational map:

$$\pi_x \times \mathrm{pr}_2 \times \cdots \times \mathrm{pr}_k \colon \mathbf{P} \dashrightarrow \mathbf{P}^{m_1-1} \times \mathbf{P}^{m_2} \times \cdots \times \mathbf{P}^{m_k} \simeq_{\mathrm{bir}} \mathbf{P}^n.$$

Let $\phi_0 = \pi_x \times \mathrm{pr}_2 \times \cdots \times \mathrm{pr}_k|_X$. If x is chosen generally then $x \times \mathbf{P}^{m_2} \times \cdots \times \mathbf{P}^{m_k} \not\subset X$. It follows that $\deg(\phi_0) = d_1 = d$. □

Now assume for contradiction that there is a dominant rational map

$$\phi \colon X \dashrightarrow \mathbf{P}^n$$

with $\deg(\phi) = \delta < d$. Let $\mathbf{P} \subset \mathbf{P}^N$ be the Segre embedding of $\mathbf{P}$ defined by $|\mathcal{O}_{\mathbf{P}}(1, \ldots, 1)|$.

Lemma 6.2. *The fibers of ϕ lie on lines in $\mathbf{P}^N$ which are contained in $\mathbf{P}$.*

Proof. The proof is the same as the proof of Lemma 2.4. We just remark that

$$\omega_X \cong \mathcal{O}_X(d_1 - m_1 - 1, \ldots, d_k - m_k - 1)$$

is p-very ample, and that $\mathbf{P} \subset \mathbf{P}^N$ is cut out by quadrics. □

Remark 6.3. For any curve $C \subset \mathbf{P}$, we have

$$\deg(\mathcal{O}_C(1,\dots,1)) = \sum \deg(\mathrm{pr}_i^*(\mathcal{O}_{\mathbf{P}}^{m_i}(1))|_C).$$

It follows that any line $\ell \subset \mathbf{P} \subset \mathbf{P}^N$ has a unique nonconstant projection $\mathrm{pr}_i\colon \mathbf{P}\to\mathbf{P}^{m_i}$. Thus the Fano variety of lines in $\mathbf{P}$ is a disjoint union:

$$\begin{aligned}\mathrm{Fano}(\mathbf{P}) = \mathrm{Gr}(2, m_1+1) \times \mathbf{P}^{m_2} \times \cdots \times \mathbf{P}^{m_k} \sqcup \cdots \sqcup \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_{k-1}} \\ \times \mathrm{Gr}(2, m_k+1).\end{aligned}$$

Now by Lemma 6.2, the map $\phi\colon X \dashrightarrow \mathbf{P}^n$ induces a rational map

$$\mathbf{P}^n \dashrightarrow \mathrm{Fano}(\mathbf{P}).$$

Assume without loss of generality that the image of $\mathbf{P}^n$ is contained in $\mathrm{Gr}(2, m_1 + 1) \times \mathbf{P}^{m_2} \times \cdots \times \mathbf{P}^{m_k}$. To simplify notation set:

$$\mathbf{P}_0 := \mathbf{P}^{m_2} \times \cdots \times \mathbf{P}^{m_k}.$$

Then as in §2 or §5 we arrive at the following fundamental diagram:

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d, hook] \\ \mathbf{P} \arrow[u, hookleftarrow] & F \arrow[l, "\pi"'] \arrow[d, "\psi"] \\ & B \arrow[r, "f"] & \mathrm{Gr}(2, m_1+1) \times \mathbf{P}_0. \end{tikzcd} \tag{9}$$

As usual $f\colon B\to\mathrm{Gr}(2, m_1 + 1) \times \mathbf{P}_0$ is a resolution of the rational map $\mathbf{P}^n \dashrightarrow \mathrm{Gr}(2, m_1 + 1) \times \mathbf{P}_0$. The map $\psi\colon F\to B$ is the family of lines in $\mathbf{P}$ parameterized by B and the map $\psi\colon F\to\mathbf{P}$ is the natural map. Finally, X' is the closure of the image of the rational map

$$\mathrm{id} \times \phi\colon X \dashrightarrow F \subset \mathbf{P} \times B,$$

in particular $X'\to X$ is birational.

Lemma 6.4. *If $p \geq m$ then the map $\pi\colon F\to\mathbf{P}$ is birational, i.e. the map ϕ determines a "congruence of lines of order one" on $\mathbf{P}$.*

Proof. The proof is similar to the proof of Lemma 2.5 but is more delicate. Again the goal is to show that $\pi_*(\pi^*([X])) = [X]$ and again we start by writing:

$$\pi^*(X) = aX' + \sum a_i E_i$$

where $a, a_i > 0$. Again it suffices to show that $a = 1$ and $\pi_* E_i = 0$.

To prove that $a = 1$ we again note that for a line $\ell \subset F$ which is a fiber of ψ, we have $\pi^*(X) \cdot [\ell] = d_1$, and $X' \cdot [\ell] = \delta$. In particular, as in Lemma 2.5 it suffices to show that $\delta > d_1/2$ and $\deg(\psi|_{E_i}) > d_1/2$.

If $x \in B$ is a general point then we know that $\#(\varphi^{-1}(x)) = \delta$ and $\varphi^{-1}(x) \subset \mathbf{P}$ satisfies Cayley–Bacharach with respect to the linear series $|\mathcal{O}_{\mathbf{P}}(d_1 - m_1 - 1, \dots, d_k - m_k - 1)|$. As $\varphi^{-1}(x)$ actually lies in a linear subspace $\mathbf{P}^{m_1} \times y$ for some $y \in \mathbf{P}_0$ we can conclude that $\varphi^{-1}(x)$ satisfies Cayley–Bacharach with respect to the restriction of the linear series:

$$\left|(\mathcal{O}_{\mathbf{P}}(d_1 - m_1 - 1, \dots, d_k - m_k - 1)|_{\mathbf{P}^{m_1} \times y})\right| = |\mathcal{O}_{\mathbf{P}^{m_1}}(d_1 - m_1 - 1)|$$

Thus applying Lemma 1.8 and our degree assumption we have that $\delta > d_1/2$. And as in Lemma 2.5, if we assume for contradiction that $\pi_* E_i \neq 0$ we similarly have $\deg(\psi|_{E_i}) > d_1/2$, which gives a contradiction. □

Finally, we need the following result about uniruled subvarieties of X.

Lemma 6.5. *Let $X \in |\mathcal{O}_{\mathbf{P}}(d_1, \dots, d_k)|$ as above be a very general divisor with all the $d_i > 1$. If S is an e-dimensional subvariety swept out by rational curves, then*

$$n \geq e + p + 1.$$

Proof. We have $\omega_X(-n, \dots, -n)$ is q-very ample for $q = p - n$. Let $L = \mathcal{O}_{\mathbf{P}}(d_1, \dots, d_k)$. Thus by Proposition 2.8 and Lemma 2.9, if we show that $\mathcal{M}_L(1, \dots, 1)$ is globally generated and

$$H^1(X, \mathcal{M}_L(1, \dots, 1)) = 0$$

then we can conclude that

$$1 \geq e + q + 2 = e + p - n + 2,$$

which is equivalent to the desired inequality.

To prove global generation $\mathcal{M}_L(1, \dots, 1)$, note that there is a surjection

$$\bigoplus_{i=1}^{k} \left(\left(\bigotimes_{j \neq i} H^0(\mathcal{O}_{\mathbf{P}^{m_j}}(d_j)) \right) \otimes_{\mathbb{C}} \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(0,\dots,d_i,\dots,0)} \right) \to \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(d_1,\dots,d_k)}.$$

If we twist this map by $\mathcal{O}_{\mathbf{P}}(1, \dots, 1)$ then the left hand side is globally generated by Proposition 2.10. Therefore $\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(d_1,\dots,d_k)}(1, \dots, 1)$ is globally generated. The vanishing $H^1(X, \mathcal{M}_L(1, \dots, 1)) = 0$ follows from the Künneth formula and a relatively straightforward diagram chase. □

Proof of Theorem E. Assume for contradiction that there is a rational map $\phi \colon X \dashrightarrow \mathbf{P}^n$ with

$$\delta := \deg(\phi) < d.$$

By Lemma 6.2 the fibers of ϕ lie on lines inside $\mathbf{P}$. Using the notation of Remark 6.3, assume without loss of generality that the lines spanned by the fibers of ϕ are nonconstant only along the projection to $\mathbf{P}^{m_1}$.

The case $m_1 = 1$ is distinct from the other cases. In this case, $\mathrm{Gr}(2, m_1 + 1)$ is a point and the map

$$f : B \to \mathrm{Gr}(2, m_1 + 1) \times \mathbf{P}_0 \cong \mathbf{P}_0$$

is birational. It is easy to deduce that the map ϕ rationally factors through the projection $X \to \mathbf{P}_0$, which has degree $\geq d$, a contradiction.

Now assume that $m_1 \geq 2$. Every variety in (9) admits a projection to $\mathbf{P}_0$, and all of the maps in (9) commute with this projection. This allows us to base change the diagram (9) to consider fibers over a very general point $y \in \mathbf{P}_0$, which gives the following diagram:

$$\begin{tikzcd}
X_y & X'_y \arrow[l] \arrow[d, hook] & \\
\mathbf{P}^{m_1} \arrow[u, hookleftarrow] & F_y \arrow[l, "\pi_y"'] \arrow[d, "\psi_y"] & \\
& B_y \arrow[r, "f"] & \mathrm{Gr}(2, m_1 + 1).
\end{tikzcd} \tag{10}$$

Because y is general, every variety in (10) is reduced and irreducible, both π_y and $\pi_y|_{X'_y}$ are birational maps, and the degree of $\psi|_{X'_y}$ is still δ.

As X was chosen to be very general, X_y is a very general degree d_1 hypersurface in $\mathbf{P}^{m_1}$ with a degree $\delta < d \leq d_1$ rational map

$$\phi_y : X_y \dashrightarrow B_y.$$

Now F_y is rational as π_y is a birational map. Moreover F_y is a $\mathbf{P}^1$ bundle over B_y, so B_y is rationally connected. The proof of [4, Thm. C] works for dominant rational maps to any rationally connected base. Thus as $\delta < d_1$ and X_y is very general in $\mathbf{P}^{m_1}$, then [4, Thm. C] implies that $\delta = d_1 - 1$, ϕ_y is projection from a point $x \in X_y$, and B_y is actually rational.

Returning to diagram (9), the centers of the projections $x \in X_y$ allow us to define a section of the generically finite map

$$\pi^{-1}(X) \to B.$$

I.e. there is a component E in $\pi^{-1}(X)$, which is different from X' such that $\psi|_E : E \to B$ has degree 1 (and $\pi(E)$ dominates $\mathbf{P}_0$). Thus $\pi(E)$ is a subvariety of X of dimension $n + 1 - m_1$ swept out by rational curves. Thus by Lemma 6.5 we see that

$$n \geq n + 1 - m_1 + p + 1,$$

which implies

$$m \geq m_1 \geq p + 2 > p$$

which contradicts the assumption that $p \geq m$. $\square$

7. Open problems

There are many possibilities for future work. We would like to pose several problems which seem like natural extensions of this paper.

First, let Z be a smooth Fano threefold and let L be an ample line bundle on Z. Assume that L is *sufficiently positive* in an appropriate sense.

Problem 7.1. Compute $\mathrm{irr}(X)$ for $X \in |L|$ any smooth surface.

The results in this paper, as well as the results from [1, Thm. 1.3], can be used to compute the degree of irrationality of every sufficiently positive smooth surface in $\mathbf{P}^3$, $\mathbf{P}^2 \times \mathbf{P}^1$, $(\mathbf{P}^1)^3$, any smooth quadric threefold, any smooth cubic threefold, or any smooth (2,2)-complete intersection threefold. In each case the degree of irrationality can be controlled by the geometry of low degree curves contained in X. A natural next step would be to compute the degree of irrationality of smooth surfaces in smooth quartic threefolds $Z \subset \mathbf{P}^4$, or smooth surfaces in quartic double solids.

It is also natural to ask how the degree of irrationality behaves in families. I.e. assume that

$$\pi : \mathcal{X} \to T$$

is a smooth family of complex varieties with relative dimension n. How does the function

$$t \in T \mapsto \mathrm{irr}(\pi^{-1}(t)) \in \mathbb{Z}$$

behave? When $n = 1$ it is well-known that the gonality of a curve is lower-semicontinuous in families. On the other hand, recently Hasset, Pirutka, and Tschinkel ([8]) constructed a family of varieties such that $\mathrm{irr}(\pi^{-1}(t))$ equals 1 on a dense set (i.e. $\pi^{-1}(t)$ is rational) but is strictly greater then 1 at the very general point $t \in T$. In a positive direction, Kontsevich and Tschinkel [9] have proved that rationality specializes in smooth projective families. The analogue for degree of irrationality is

Question 7.2. Let $\pi : \mathcal{X} \to T$ be a smooth family of complex projective varieties over a curve T. Can the function $\mathrm{irr}(\pi^{-1}(t))$ only decrease upon specialization?

Finally, we ask if there is a more general Cayley–Bacharach result. Let $\mathcal{S}$ be a set of r points in projective space which satisfy the Cayley–Bacharach condition with respect to $|mH|$.

Question 7.3. If $r \leq \left(\frac{d+3}{2}\right) m + 1$, is $\mathcal{S}$ contained in a degree d curve in projective space?

Here the ratio $(d + 3)/2$ should be thought of as the ratio between the number of general points that degree d plane curves can interpolate $\binom{d+2}{2} - 1$ and the degree d. Presumably, the proof of such a result would have to be less ad hoc then our proof of Theorem 1.9.

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